Scalable fine-grained proofs for formula processing

 $^1 \rm University$ of Lorraine, CNRS, Inria, LORIA, Nancy, France $^2 \rm Vrije$ Universiteit Amsterdam, The Netherlands



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- $\,\vartriangleright\,$ to check the result for unsatisfiable/valid formulas
- \triangleright for solver/prover cooperation
- \triangleright as a debugging facility
- \triangleright for evaluation purposes (how good is the algorithm?)
- \triangleright as a part of the reasoning framework (e.g. conflict clauses)
- \triangleright to extract cores
- \triangleright to compute interpolants

Challenges for proofs in FOL

 $\,\vartriangleright\,$ Collecting and storing proof information efficiently

 \triangleright Producing proofs for sophisticated processing techniques

> Producing proofs for modules that use external tools

▷ Standardizing a proof format

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- Collecting and storing proof information efficiently no convergence, but quite active [KBT+16; HBR+15; KV13; Sch13; BODF09; WDF+09; Mos08; MB08; SZS04]
- Producing proofs for sophisticated processing techniques proofs with holes or too coarse
- Producing proofs for modules that use external tools depends on tool arbitrarily complex to reconstruct information
- Standardizing a proof format open

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Resolution chains, input formulas, tautologies for theory and quantifier reasoning

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 \triangleright SAT solver: resolution

$$\frac{A \lor \ell}{A \lor B} \xrightarrow{B \lor \overline{\ell}}$$

 $\begin{array}{l} \text{Antecedents:} \ A \lor \ell, \ B \lor \overline{\ell} \\ \text{Pivot:} \ \ell \ \text{or} \ \overline{\ell} \\ \text{Resolvent:} \ A \lor B = (A \lor \ell) \diamond (B \lor \overline{\ell}) \end{array}$

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 \triangleright theory solvers: theory lemmas

$$\neg (a \simeq c) \lor \neg (c \simeq b) \lor a \simeq b \qquad \neg (a \simeq b) \lor f(a) \simeq f(b)$$
$$\neg (y > 1) \lor \neg (x < 1) \lor y > x$$

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$$\begin{aligned} \neg(a\simeq c) \lor \neg(c\simeq b) \lor a\simeq b & \neg(a\simeq b) \lor f(a)\simeq f(b) \\ \neg(y>1) \lor \neg(x<1) \lor y>x \end{aligned}$$

▷ instantiation module: instantiation lemmas

$$\neg(\forall x.\,\psi[x]) \lor \psi[t]$$

What is hard about proofs for formula processing?

Resolution does not capture all transformations

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Code is lengthy and deals with many cases

Proving formula processing

Extensible framework to represent proofs for processing techniques involving *locally replacing equals by equals* in the presence of *binders*

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Some instances:

Skolemization: $(\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))$

let elimination: (let $x \simeq a$ in p(x, x)) $\simeq p(a, a)$

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Challenge is to manipulate bound variables and substitutions soundly and efficiently

Inference system

A context Γ fixes a set of variables and specifies a substitution

$$\Gamma ::= \varnothing \mid \Gamma, x \mid \Gamma, \bar{x}_n \mapsto \bar{s}_n$$
 bound variable

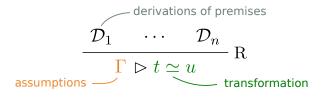
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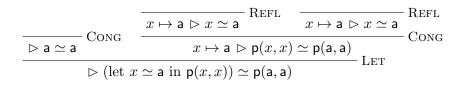
Rules have the form

b

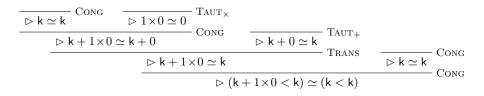


 $\vartriangleright\,$ Semantically, the judgment expresses the equality of the terms $\Gamma(t)$ and u for all variables fixed by Γ

Example of 'let' expansion



Example of theory simplification



Output skolemization

The skolemization proof of the formula $\neg \forall x. p(x)$:

$$\frac{\overline{x \mapsto \varepsilon x. \neg \mathbf{p}(x) \triangleright x \simeq \varepsilon x. \neg \mathbf{p}(x)}}{x \mapsto \varepsilon x. \neg \mathbf{p}(x) \triangleright \mathbf{p}(x) \simeq \mathbf{p}(\varepsilon x. \neg \mathbf{p}(x))} \operatorname{Cong}_{\operatorname{Ko}_{\forall}} \frac{\nabla (\forall x. \mathbf{p}(x)) \simeq \mathbf{p}(\varepsilon x. \neg \mathbf{p}(x))}{\nabla (\neg \forall x. \mathbf{p}(x)) \simeq \neg \mathbf{p}(\varepsilon x. \neg \mathbf{p}(x))} \operatorname{Cong}_{\operatorname{Cong}}$$

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The skolemization proof of the formula $\neg \forall x. p(x)$:

$$\frac{\overbrace{x \mapsto \varepsilon x. \neg p(x) \triangleright x \simeq \varepsilon x. \neg p(x)}^{\text{REFL}}}{\overbrace{x \mapsto \varepsilon x. \neg p(x) \triangleright p(x) \simeq p(\varepsilon x. \neg p(x))}^{\text{REFL}}} \xrightarrow{\text{Cong}} \frac{}{} \sum (\forall x. p(x)) \simeq p(\varepsilon x. \neg p(x))} \xrightarrow{\text{SKO}_{\forall}} \frac{}{} \sum (\forall x. p(x)) \simeq p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \frac{}{} \sum (\forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \frac{}{} \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \sum (\neg \forall x. p(x)) \simeq \neg p(\varepsilon x. \neg p(x))} \xrightarrow{\text{Cong}} \sum \sum (\neg \forall x. p(x)) \xrightarrow{\text{Cong}} \sum (\neg \forall x. p($$

veriT syntax:

$$\begin{array}{ll} (.c0 \; (\mathsf{Sko_All} \; : \operatorname{conclusion} \; ((\forall x. \; \mathsf{p}(x)) \simeq \mathsf{p}(\varepsilon x. \; \neg \; \mathsf{p}(x))) \\ & : \operatorname{args} \; (x \mapsto (\varepsilon x. \; \neg \; \mathsf{p}(x))) \\ & : \operatorname{subproof} \; ((.c1 \; (\mathsf{Refl} \; : \operatorname{conclusion} \; (x \simeq (\varepsilon x. \; \neg \; \mathsf{p}(x))))) \\ & \quad (.c2 \; (\mathsf{Cong} \; : \operatorname{clauses} \; (.c1) \\ & \quad : \operatorname{conclusion} \; (\mathsf{p}(x) \simeq \mathsf{p}(\varepsilon x. \; \neg \; \mathsf{p}(x))))))) \\ (.c3 \; (\mathsf{Cong} \; : \operatorname{clauses} \; (.c0) \; : \operatorname{conclusion} \; ((\neg \; \forall x. \; \mathsf{p}(x)) \simeq \neg \; \mathsf{p}(\varepsilon x. \; \neg \; \mathsf{p}(x)))))) \end{array}$$

Proof-producing contextual recursion

```
function process(\Delta, t)
   match t
      case x:
         return build_var(\Delta, x)
      case f(\bar{t}_n):
         \bar{\Delta}'_n \leftarrow (ctx_{app}(\Delta, f, \bar{t}_n, i))_{i=1}^n
         return build_app(\Delta, \bar{\Delta}'_n, f, \bar{t}_n, (process(\Delta'_i, t_i))_{i=1}^n)
      case Qx. \varphi:
         \Delta' \leftarrow \mathsf{ctx}_\mathsf{quant}(\Delta, Q, x, \varphi)
         return build_quant(\Delta, \Delta', Q, x, \varphi, \text{ process}(\Delta', \varphi))
      case let \bar{x}_n \simeq \bar{r}_n in t':
         \Delta' \leftarrow \mathsf{ctx}_{\mathsf{let}}(\Delta, \bar{x}_n, \bar{r}_n, t')
         return build_let(\Delta, \Delta', \bar{x}_n, \bar{r}_n, t', process(\Delta', t'))
```

> Parameterized by a notion of context and plugin functions

Soundness of inference rules proven through an encoding into simply typed $\lambda\text{-calculus}$

Correctness of proof-producing contextual recursion algorithm

Cost of proof production is linear and of proof checking is (almost) linear* * assuming suitable data structures

Proof output for veriT

Framework implemented with a proof-producing contextual recursion algorithm

- \oplus fine-grained proofs for most processing transformations
- \oplus No negative impact on performance
- \oplus More transformations in proof producing mode
- $\oplus \$ Dramatic simplification of the code base

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Prototype checker in Isabelle/HOL

Maps proofs into Isabelle theorems

 \oplus Judgements encoded in λ -calculus

Conclusions

- \triangleright Centralizes manipulation of bound variables and substitutions
- ▷ Accommodates many transformations (e.g. Skolemization)
- ▷ Proof checking is (almost) linear
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Future work

- ▷ Support global rewritings within the framework
- ▷ Support richer logics (e.g. HOL)
- ▷ Implement proof reconstruction in Isabelle/HOL

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